Positive solutions for three-point boundary value problems

A. Lepin and F. Sadyrbaev

Summary. We provide the conditions for existence of positive solutions for the boundary value problem $x'' = f(t, x, x'),\ p\,x(0) + x'(0) = 0,\ x(1) = \alpha x(\eta)$.

Key words: boundary value problems, positive solutions
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1 Introduction

In the work [1] the conditions for existence of a positive solution to the boundary value problem

$$x'' + g(t)f(x, x') = 0, \quad x'(0) = 0,\ x(1) = \alpha x(\eta), \quad \alpha, \eta \in (0, 1) \tag{1}$$

were obtained. Moreover, solvability of the problem

$$\begin{aligned} &x'' = g(t, x, x') + h(t, x, x'), \quad t \in I := [0, 1], \\ &px(0) + x'(0) = 0, \quad x(1) = \alpha x(\eta), \quad \alpha \le 0,\ 0 < \eta < 1 \end{aligned} \tag{2}$$

has been proved under the assumptions

$$\begin{aligned} &x' g(t, x, x') \le 0, \\ &|h(t, x, x')| \le a(t)|x| + b(t)|x'| + u(t)|x|^r + v(t)|x'|^k + e(t), \quad 0 \le r, k < 1, \\ &(|p| + a_1)e^{b_1} < 1, \quad a_1 = \|a\|_1 = \textstyle\int_0^1 |a(t)|\, dt, \quad b_1 = \|b\|_1. \end{aligned}$$

Similar boundary value problems were considered in the works [2] - [11].

Our purpose in this paper is twofold. First, we prove the existence of a positive solution for more general problem than the problem (1). Second, we will show that the problem (2) is solvable also if $(p_+ + a_1)e^{b_1} < 1$, where $p_+ = \max\{0, p\}$. Examples show that these conditions cannot be improved.

2 Existence of positive solutions

Consider the problem

$$\begin{aligned} &x'' = f(t,x,x'), \quad px(0)+x'(0)=0,\ x(1)=\alpha x(\eta), \\ &p \in C([0,+\infty),\mathbb{R}), \quad \alpha,\eta \in (0,1). \end{aligned} \tag{3}$$

where $f : I \times [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, that is, (i) $f(\cdot,x,y)$ is measurable in I for fixed $x,y \in \mathbb{R}$; (ii) $f(t,\cdot,\cdot)$ is continuous in $\mathbb{R}^2$ for a.e. $t \in I$; (iii) for any compact set $P \subset \mathbb{R}^2$ there exists a function $g \in L_1(I,\mathbb{R})$ such that for any $(t,x,y) \in I \times P$ the inequality $|f(t,x,y)| \le g(t)$ holds.

Suppose that the following conditions hold:

(1) There exist functions $a,b,c,d \in L_1(I,[0,+\infty))$ such that for any $\varepsilon > 0$ and some $e \in L_1(I,[0,+\infty))$ the relation

$$\begin{aligned} &f(t,x,x') \ge -(a(t)+\varepsilon c(t))x + (b(t)+\varepsilon d(t))x' - e(t), \\ &(t,x,x') \in I \times [0,+\infty) \times (-\infty,0]; \end{aligned}$$

(2) For any $\tau \in (0,1]$ boundedness of a solution $x_N : [0,\tau) \to \mathbb{R}$ to the Cauchy problem

$$x'' = f(t,x,x'), \quad x(0)=N,\ x'(0) = -p(N),\ N>0 \tag{4}$$

implies boundedness of the derivative $x'_N(t)$;

(3) There exists $\delta > 0$ such that

$$f_*(t) = \max\{f(t,x,x') : 0 \le x \le \delta,\ -\delta \le x' \le \delta\} \le 0, \quad t \in I,\ \|f_*\|_1 > 0;$$

(4) $p(0) \ge 0$;

(5) There exist $p_+, q \in [0,+\infty)$ such that $p(x) \le p_+x + q,\ x \ge 0$.

Remark: The condition 2 holds if for $(t,x,x') \in I \times [0,+\infty) \times [0,-\infty)$

$$f(t,x,x') \le (a(t)+\varepsilon c(t))x + (b(t)+\varepsilon d(t))x' + e(t).$$

Lemma 2.1 *Let* $A,B,E \in L_1(I,[0,+\infty))$, $N>0$ *and*

$$f(t,x,x') \ge -A(t)x + B(t)x' - E(t), \quad (t,x,x') \in I \times [0,+\infty) \times (-\infty,0].$$

If the condition

$$(p_+ + A_1 + \frac{E_1+q}{N})e^{B_1} < 1$$

holds, where $A_1 = \|A\|_1,\ B_1 = \|B\|_1,\ E_1 = \|E\|_1$, *then a solution* $x_N : I \to [0,+\infty)$ *to the Cauchy problem*

$$x'' = f(t,x,x'), \quad x(0)=N,\ x'(0) = -p(N)$$

satisfies the estimates

$$x'_N \ge -(N_mp_+ + N_mA_1 + E_1 + q)e^{B_1}, \tag{5}$$

$$x_N(1) \ge N_m - (N_mp_+ + N_mA_1 + E_1 + q)e^{B_1}, \tag{6}$$

$$\frac{x_N(1)}{x_N(\eta)} \ge \frac{1-(p_+ + A_1 + \frac{E_1+q}{N_m})e^{B_1}}{1-\eta(p_+ + A_1 + \frac{E_1+q}{N_m})e^{B_1}}, \tag{7}$$

where $N_m = \max\{x_N(t) :\ t \in I\}$.

Proof. Consider the case $x'_N \le 0$. Then $N_m = N$,

$$x''_N = f(t, x_N, x'_N) \ge -A(t)x_N + B(t)x'_N - E(t) \ge B(t)x'_N - A(t)N - E(t)$$

and $x'_N(0) = -p(N) \ge -p_+N - q$. Let y_N be a solution to the Cauchy problem

$$y' = B(t)y - A(t)N - E(t), \quad y(0) = -p_+N - q. \tag{8}$$

Comparison theorems for the first order differential inequalities imply that $x'_N \ge y_N$. A solution to the Cauchy problem (8) has the form

$$\begin{aligned} y_N(t) = \; & -(p_+N + q)\exp\left(\int_0^t B(s)\,ds\right) \\ & - \int_0^t (A(s)N + E(s))\exp\left(\int_s^t B(\xi)\,d\xi\right) ds. \end{aligned} \tag{9}$$

It follows from (9) that

$$x'_N \ge y_N \ge -(p_+N + q + NA_1 + E_1)e^{B_1}. \tag{10}$$

Consider the case $x_N \le N$. Let $T = \{t \in (0,1) : \ x'_N(t) < 0\}$. It is clear that T is an open set which can be represented as a union of disjointed open intervals. Denote a sample interval (t_1, t_2). In case of $t_1 = 0$ the estimate $x'_N(t) \ge y_N(t), \ t \in [t_1, t_2]$ can be obtained as above. If $t_1 > 0$, then $x'_N(t_1) = 0$ and the estimate $x'_N(t) \ge y_N(t), \ t \in [t_1, t_2]$ can be obtained also. The inequality $x'_N(t) \ge y_N(t)$ is evident if $x'_n(t) \ge 0$. Now (10) implies (5) which, in turn, implies (6).

Consider the case $N_m > N$. As before one obtains the relations

$$x'_N \ge -(N_mA_1 + E_1)e^{B_1} \ge -(N_mp_+ + N_mA_1 + E_1 + q)e^{B_1},$$

$$x_N(1) \ge N_m - (1-\tau)(N_mA_1 + E_1)e^{B_1} \ge N_m - (N_mp_+ + N_mA_1 + E_1 + q)e^{B_1},$$

where $N_m = x_N(\tau)$. The estimate (7) follows from

$$\begin{aligned} \frac{x_N(1)}{x_N(\eta)} &\ge \frac{x_N(1)}{x_N(1) + (1-\eta)(N_mp_+ + N_mA_1 + E_1 + q)e^{B_1}} \\ &= 1 - \frac{(1-\eta)(N_mp_+ + N_mA_1 + E_1 + q)e^{B_1}}{x_N(1) + (1-\eta)(N_mp_+ + N_mA_1 + E_1 + q)e^{B_1}} \\ &\ge 1 - \frac{(1-\eta)(N_mp_+ + N_mA_1 + E_1 + q)e^{B_1}}{N_m - (N_mp_+ + N_mA_1 + E_1 + q)e^{B_1} + (1-\eta)(N_mp_+ + N_mA_1 + E_1 + q)e^{B_1}} \\ &= \frac{1 - (p_+ + A_1 + \frac{E_1+q}{N_m})e^{B_1}}{1 - \eta(p_+ + A_1 + \frac{E_1+q}{N_m})e^{B_1}}. \end{aligned}$$

Theorem 2.1 *If $(p_+ + a_1)e^{b_1} < 1$, then there exists a positive solution to the problem (3) for*

$$\alpha \in \left(0, \frac{1 - (p_+ + a_1)e^{b_1}}{1 - \eta(p_+ + a_1)e^{b_1}}\right). \tag{11}$$

Proof. Let us fix η and α, which satisfy the condition (11). Define f for $x<0$ as

$$f(t,x,x')=f(t,0,x'),\quad (t,x,x')\in I\times(-\infty,0)\times\mathbb{R}.$$

Consider a solution of the Cauchy problem (4) for sufficiently small $N=N_1$. It follows from the conditions 3 and 4 that the graph of x_{N_1} crosses the t-axis and decreases for t such that $x_{N_1}<0$. Choose $\varepsilon>0$ so that the inequalities

$$\begin{array}{l}(p_+ + a_1+\varepsilon c_1)e^{b_1+\varepsilon d_1}<1,\quad c_1=\|c\|_1,\ d_1=\|d\|_1,\\ \alpha<(1-(p_+ + a_1+\varepsilon c_1)e^{b_1+\varepsilon c_1})/(1-\eta(p_+ + a_1+\varepsilon c_1)e^{b_1+\varepsilon c_1})\end{array}\tag{12}$$

are satisfied. Using the condition 1 for a given ε find $e(t)$ and choose $N_2>0$ so that the the inequalities

$$\begin{array}{l}(p_+ + a_1+\varepsilon c_1+\frac{e_1+q}{N_2})e^{b_1+\varepsilon d_1}<1,\quad e_1=\|e\|_1,\\ \alpha<(1-(p_+ + a_1+\varepsilon c_1+\frac{e_1+q}{N_2})e^{b_1+\varepsilon c_1})/(1-\eta(p_+ + a_1+\varepsilon c_1+\frac{e_1+q}{N_2})e^{b_1+\varepsilon c_1})=\alpha_{N_2}.\end{array}$$

are satisfied. We wish now to show that the boundary value problem

$$x''=f(t,x,x'),\quad p(x(0))+x'(0)=0,\ x(1)=N_2\tag{13}$$

is solvable. For this, let us consider the modified equation

$$x''=f_{LM}(t,x,x'):=f(t,\delta(-L,x,L),\delta(-M,x',M)),\tag{14}$$

where $L,M\in(0,+\infty)$, together with the boundary conditions in (13). The function $\delta(u,v,w)=u$ for $v<u$, $\delta=v$ for $u\le v\le w$, and $\delta=w$ for $v>w$. Notice that the conditions 1 to 3 are satisfied also for f_{LM}. Let $N_3\in(N_2,+\infty)$ be such that

$$N_3-N_3(p_+ + a_1+\varepsilon c_1+\frac{e_1+q}{N_3})e^{b_1+\varepsilon}>N_2.\tag{15}$$

Lemma 1 applied to the problem

$$x''=f_{LM}(t,x,x'),\quad x(0)=N_3,\ x'(0)=-p\,(N_3)\tag{16}$$

yields the estimate for the solution

$$\begin{array}{rl}x_{LMN_3}(1) & \ge N_m-(N_m p_+ + N_m a_1+N_m\varepsilon c_1+e_1+q)e^{b_1+\varepsilon d_1}\\ & \ge N_3-(N_3 p_+ + N_3 a_1+N_3\varepsilon c_1+e_1+q)e^{b_1+\varepsilon d_1}>N_2.\end{array}\tag{17}$$

A set of solutions to the problem

$$x''=f_{LM}(t,x,x'),\quad x(0)=N,\ x'(0)=-p\,(N),\quad N\in[N_1,N_3]\tag{18}$$

is connected. Therefore there exist $N_0,N_4\in[N_1,N_3]$ and solutions x_{LMN_0},x_{LMN_4} of the problem (18), where $N=N_0$ and $N=N_4$ respectively such that $x_{LMN_0}(1)=0$ and $x_{LMN_4}(1)=N_2$. Note that $x_{LMN_0}\ge 0$ and $x_{LMN_4}>0$. Lemma 1 implies the existence of $L_1>0$ such that for any solution x_{LMN} of the Cauchy problem (8) from the estimate $0\le$

$x_{LMN}(1) \le N_2$ follows $x_{LMN}(1) \le L_1$. Let $0 \le x_{LMN}(1) \le N_2$ hold. Then $x_{MN} = x_{L_1MN}$ is a solution of the Cauchy problem

$$x'' = f_M(t,x,x') := f(t,x,\delta(-M,x',M)), \quad x(0) = N, \quad x'(0) = -p(N). \tag{19}$$

Let us show that there exists $M_* > 0$ such that the estimate $0 \le x_{MN}(1) \le N_2$ implies $|x'_{MN}| < M_*$. Suppose the contrary is true. Set $M_0 = \max\{|p(N)| : N \in [N_1, N_3]\}$ and let a sequence $M_n \in (M_0, +\infty)$, $n = 5, 6, \dots$ tend to $+\infty$. Then there exists a sequence of solutions $x_n = x_{M_nN_n}$ and $t_n \in (0,1]$ such that

$$0 \le x_n(1) \le N_2, \quad |x'_n(t)| < M_n, \ t \in [0, t_n), \quad |x'_n(t_n)| = M_n.$$

A subsequence exists which converges to a solution of the Cauchy problem (4), which does not satisfy the condition 2. Thus the estimate $|x'_{MN}| < M_*$ is valid. Therefore $x_{N_0} = x_{M_*N_0}$, $x_{N_4} = x_{M_*N_4}$ are solutions of the boundary value problem (4) for $N = N_0$ and $N = N_4$ respectively and satisfy the conditions $x_{N_0}(1) = 0$, $x_{N_4}(1) = N_2$. Notice that $x_{N_0}(1)/x_{N_0}(\eta) = 0$ and $x_{N_4}(1)/x_{N_4}(\eta) \ge \alpha_{N_2} > \alpha$ by virtue of Lemma 1.

Let us made an extra assumption that solutions of the Cauchy problems (4) are defined uniquely. Let N_4 be a minimal value of $N \in [N_1, +\infty)$ for which $x_N(1) = N_2$ and let N_0 be a maximal value of $N \in [N_1, N_4]$ for which $x_{N_0}(1) = 0$. A set of solutions of the Cauchy problems (4) for $N \in [N_0, N_4]$ is connected. Hence there exists $N_\alpha \in [N_0, N_4]$ such that $x_{N_\alpha}(1)/x_{N_\alpha}(\eta) = \alpha$. Evidently x_{N_α} solves the BVP (3). It follows from the condition 3 that $x_{N_\alpha} > 0$.

The extra assumption above will now be eliminated by approximation arguments. We do not assume now that solutions of the Cauchy problems (4) are defined uniquely. Let $t \in I$ be fixed. Consider the mesh

$$x_{ni} = i\delta 2^{-n}, \quad x'_{nj} = j\delta 2^{-n}, \quad i,j = \dots, -1, 0, 1, \dots, \quad n = 1, 2, \dots$$

in the (x,x')-plane. Let n be fixed. Substitute the function $f(t,\cdot,\cdot)$ on the triangle with vortices (x_{ni}, x'_{nj}), (x_{ni+1}, x'_{nj}) and (x_{ni+1}, x'_{nj+1}) by a plane which coincides with $f(t,\cdot,\cdot)$ at these points. Similarly we approximate the function f on the triangle with vortices (x_{ni}, x'_{nj}), (x_{ni}, x'_{nj+1}) and (x_{ni+1}, x'_{nj+1}). Denote the approximating function by $f_n(t,x,x')$. The function f_n satisfies the generalized Lipschitz condition and meets the hypotheses 1 and 3. It follows from the above arguments that the boundary value problems

$$x'' = f_{nLM}(t,x,x') = f_n(t, \delta(-L,x,L), \delta(-M,x'M)),$$
$$p(x(0)) + x'(0) = 0, \ x(1) = \alpha x(\eta)$$

have solutions x_{nLM} such that $0 < x_{nLM}(1) < N_2$ and $0 < x_{nLM}(1) < L_1$. Hence the function $x_{nM} = x_{nL_1M}$ solves the boundary value problem

$$x'' = f_{nM}(t,x,x') = f_n(t,x,\delta(-M,x'M)),$$
$$p(x(0)) + x'(0) = 0, \ x(1) = \alpha x(\eta).$$

If a constant $M_* > 0$ exists such that the estimates $|x'_k| < M_*$, $\ k = 1, 2, \dots$ hold for the sequence $x_k = x_{n_kM_*}$, $\ k = 1, 2, \dots$, then a subsequence can be extracted from the sequence$\{x_k\}$, which converges to a solution of the boundary value problem (3). Suppose the contrary is true. Then as above one can find $t_k \in (0,1]$ and a sequence of solutions $x_k = x_{n_kM_k}$ such that $|x'_k(t)| < M_k$, $\ t \in [0, t_k)$ and $|x'_k(t_k)| = M_k$. This sequence contains a subsequence which converges to a solution of the Cauchy problem (4), which does not meet the condition 2.

Remark 2. It follows from the proofs of Lemma 1 and Theorem 1 that q may be a function $q : [0, +\infty) \to [0, +\infty)$, satisfying the condition $\lim_{x\to+\infty} \frac{q(x)}{x} = 0$.

The result below can be proved analogously to Theorem 1.

Theorem 2.2 *If $(p_+ + a_1)e^{b_1} < 1$ and either $a_1 > 0$ or $a_1 = 0,\ b_1 > 0$ and $p_+ > 0$, then there exists a positive solution to the problem (3) for*

$$\alpha \in \left(0, \frac{1 - (p_+ + a_1)e^{b_1}}{1 - \eta(p_+ + a_1)e^{b_1}}\right). \tag{20}$$

Example 1. Consider the problem

$$x'' = f_\sigma(t, x, x'), \quad p_+x(0) + \sigma + x'(0) = 0, \quad x(1) = \alpha x(\eta), \tag{21}$$

where $\sigma \in (0, 1/3)$ and

$$\begin{aligned} &f_\sigma(t, x, x') = -a_1\sigma^{-1}x, \quad (t, x, x') \in [0, \sigma) \times [0, +\infty) \times R,\\ &f_\sigma(t, x, x') = \min\{0, b_1\sigma^{-1}x'\}, \quad (t, x, x') \in [\sigma, 2\sigma) \times [0, +\infty) \times R,\\ &f_\sigma(t, x, x') = -1, \quad (t, x, x') \in [2\sigma, 1] \times [0, +\infty) \times R. \end{aligned}$$

In case of $(p_+ + a_1)e^{b_1} < 1$ the BVP (21) shows that the conditions (11) and (20) are sharp for fixed η and sufficiently small σ.

If $(p_+ + a_1)e^{b_1} = 1$ and $a_1 = b_1 = 0$, then the BVP (21) has not a solution.

If $(p_+ + a_1)e^{b_1} = 1$ and $a_1 + b_1 > 0$, then the BVP (21) has a positive solution only for $\alpha \in (0, \alpha_\sigma)$, $\alpha_\sigma > 0$ and $\lim_{\sigma\to 0} \alpha_\sigma = 0$.

If $(p_+ + a_1)e^{b_1} > 1$, then the BVP (21) has not a solution for sufficiently small σ.

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А. Лепин, Ф. Садырбаев. Положительные решения трехточечной краевой задачи.

Аннотация. Указаны условия существования положительного решения краевой задачи $x'' = f(t, x, x')$, $p\,x(0) + x'(0) = 0$, $x(1) = \alpha x(\eta)$.

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Anotācija. Tiek doti pozitīva atrisinājuma eksistences nosacījumi robežproblēmai $x'' = f(t, x, x')$, $p\,x(0) + x'(0) = 0$, $x(1) = \alpha x(\eta)$.

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